Dynamical Systems, Ergodicity, Poincaré Recurrence, and All That

A deterministic, time translationally invariant *dynamical system* is a quadruple $(X, \mathcal{B}, \mu, \phi_t)$, where X is the state space, μ is a measure on X with $\mu(X) = 1$, $\mathcal{B}$ is the set of measurable sets of X, and for each $t \in \mathbb{R}$, $\phi_t : X \to X$ is a one-one map. It is required that the ϕ_t are measure preserving, i.e. for any $A \in \mathcal{B}$ and for any $t \in \mathbb{R}$, $\mu(\phi_t(A)) = \mu(A)$, and that they have the group property $\phi_{t_1+t_2} = \phi_{t_2} \circ \phi_{t_1}$ with $\phi_{t=0} = id$ and $\phi_{-t} = \phi_t^{-1}$. *Example*: Let $X = \mathbb{R}^{6N}$ be the usual (q, p) phase space for a Hamiltonian system. Hamilton's equations define a deterministic flow on phase space that conserves volume relative to the measure $d\mu = dqdp$ (Liouville's Theorem). To ensure that the measure normalizes, the region $X \subset \mathbb{R}^{6N}$ of the state space available to the system has to be limited so that X has compact closure. This can be guaranteed, for example, by confining the particles to a box, preventing the box from exchanging energy with its environment, and requiring that the intra-particle interaction potential is bounded from below. For such cases the relevant measure is μ "cut down" to a constant energy surface.

In this setting, *time reversal invariance* means that for every $x \in X$ and every t, $\phi_{-t}({}^R[\phi_t(x)]) = {}^R x$, where ${}^R x$ stands for the reverse of x. For Hamiltonian dynamics where $x = (q, p)$, ${}^R(q, p) = (q, -p)$.

Poincaré recurrence theorem. In a dynamical system $(X, \mathcal{B}, \mu, \phi_t)$, if $A \in \mathcal{B}$ is any measurable set, $\mu(F) = 0$ where $F := \{x \in A : \phi_t(A) \notin A \ \forall t > 0\}$.

A dynamical system is said to be *ergodic* just in case for any measurable $A \in \mathcal{B}$, if $\phi_t(A) = A \ \forall t$, then either $\mu(A) = 0$ or $\mu(A) = 1$. An equivalent characterization of ergodicity is the requirement that for any $A \in \mathcal{B}$ where $\mu(A) \neq 0$ and for almost any $x \in X$, there is a $t > 0$ such that $\phi_t(x) \cap A \neq 0$.

Lemma 1. Ergodicity implies that for any $A \in \mathcal{B}$,

$$\mu(A) = \lim_{t \to \infty} \frac{\int_0^t I_A(\phi_t(x))dt}{t}$$

for almost any $x \in X$, where $I_A(x) = 1$ for $x \in A$ and 0 otherwise.

Lemma 2. Ergodicity implies that if μ' is any invariant measure that is absolutely continuous with respect to μ (i.e. for any $A \in \mathcal{B}$, $\mu'(A) = 0$ implies that $\mu'(A) = 0$), then $\mu' = \mu$.

A measure preserving dynamical system is said to be *mixing* just in case for any $A, B \in \mathcal{B}$, $\lim_{t\to\infty} \mu(\phi_t(A) \cap B) = \mu(A) \cdot \mu(B)$. Mixing implies ergodicity but not vice versa.

The Logic of Boltzmann's Explanation of the 2nd Law

Coarse graining. Choose a set $\{m\}$ of macrostates for describing the outcomes of measurements that can be made on the system with macroscopic instruments. It is assumed that the macrostates supervene on the microstates, i.e. each m corresponds to a measurable region $M \subseteq X$ in the sense that at any time t, the system is in macrostate m just in case the microstate state x_t at t belongs to M. Close $\{m\}$ under Boolean operations to make a Boolean algebra $\{m\}^C$, and define a probability measure on $\{m\}^C$ in the obvious way: for $m \in \{m\}^C$, $\Pr(m) := \mu(M)$.

Boltzmann entropy. Assume that for each $x \in X$ there is a finest macrostate $m^f \in \{m\}^C$ actualized by the microstate x. Then (relative to the chosen coarse graining) the Boltzmann entropy $S_B(t)$ of the system at t is

$$S_B(t) := S_B(m_t^f) = k\log(\Pr(m_t^f)) = k\log(\mu(M_t^f)$$

where m_t^f is the finest macrostate actualized by the microstate at t. From here on, drop the superscript f.

The statistical version of the Second Law. What we want:

> (B) Suppose that at $t = 0$ the Boltzmann entropy $S_B(0)$ of the system is low; then for some appropriate $t_1 > 0$, it is highly probable that $S_B(t_1) > S_B(0)$.

Comments:

1) The truth of (B) depends on features of the microdynamics. In particular, for (B) to be true, ϕ_t must be such that for the overwhelming majority of microstates x in the region M_0 corresponding to the low entropy initial macrostate m_0 at $t = 0$, the macrostate m_1 at $t = t_1$ that results from the evolution $x \longmapsto \phi_{t_1}(x)$ corresponds to a region M_1 such that $\mu(M_1) >> \mu(M_0)$. The quasi-law status of (B) rests on the presumed fact that this feature of the microdynamics does obtain for the sorts of systems we subject to thermodynamic analysis and for the sorts of coarse graining relevant to explaining macroscopic observations made on these systems over time periods of length comparable to the said t_1.

2) What is the connection between the measure μ and physical probability in a propensity or a frequency sense? How does the presumed truth of (B) guarantee that there is an objective tendency for the entropy to increase? Boltzmann realized that a connection could be made between μ and the limiting relative frequency sense of probability if the system is ergodic. But are actual physical systems ergodic? Is some sort of approximate ergodicity good enough?

3) Another worry focuses on the conditions of applicability of (B) to concrete physical systems. One such condition is that for the system of interest, the microstate

at $t = 0$ will be "typical" of the microstates compatible with the observed macrostate at that time. The idea can be made more precise in the following Statistical Postulate.

> (SP) Let m_o be the macrostate at $t = 0$ of the system of interest. Then the probability at $t = 0$ that the microstate of this system lies in some measurable subset $A \subseteq M_0$ of the state space region $M_0 \subseteq X$ corresponding to m_0 is $\mu(A)/\mu(M_0)$.

4) If (SP) is true at time $t = 0$, can it be true at a later time t_1? The macrostate m_1 at t_1 is such that the microstates in M_1 that have evolved from the phase space region M_0 corresponding to the initial macrostate m_o at $t = 0$ are only a small μ-fraction of the microstates of M_1. This does not entail that the microstate at t_1 cannot be typical of the microstates compatible with the macrostate at t_1; for the sense of typicality relevant to (SP), it suffices that the microstates that have evolved from the initial macrostate are sufficiently spread over the measurable subsets of M_1. Here the property of mixing can come to the rescue–if the system is mixing.

5) Even waiving 1)-4) there are two remaining problems with which Boltzmann struggled:

a) *the initial state problem*–why was the system in a low entropy state to begin with?

b) *the time symmetry problem*–given the presumed time reversal invariance of the microdynamics, one would expect that if (B) is true, it should also be true that

> (B*) Suppose that at $t = 0$ the Boltzmann entropy $S_B(0)$ of the system is low; then for the $t_1 > 0$ of (B) it is just as probable that $S_B(-t_1) > S_B(0)$ as that $S_B(t_1) > S_B(0)$.

But if (B*) is correct to undermines the sorts of inferences we want to make about the past.

Boltzmann's attempted solutions

(1) *Cosmology to the rescue*

The entire universe began in a low entropy state, and ever since the increase in entropy has been monotonic, and the increase has been sufficiently slow as to leave the present value of entropy well below its maximum.

Problem: What is this "beginning"? Boltzmann had no good answer. Does modern Big Bang cosmology fill the bill?

(2) *Fluctuation hypothesis + anthropic principle*

"If we assume the universe great enough, we can make the probability of one relatively small part being in any given state (however far from the state of thermal equilibrium) as great as we please. We can also make the probability great that, though the whole universe is in thermal equilibrium, our world is in its present state."

Problem: Boltzmann brain paradox.

(3) *Define your way to a solution*

"For the universe as a whole, the two directions of time are indistinguishable, just as in space there is no up or down. However, just as at a particular place on the earth's surface we can call 'down' the direction toward the center of the earth, so a living being that finds itself in such a world at a certain period of time can define the time direction as going from the less probable [lower entropy] to the more probable [higher entropy] states (the former the former will be the 'past' and the latter will be the 'future') and by virtue of this definition he will find that this small region, isolated from the rest of the universe, is "initially" always in an improbable [low entropy] state."

Problem: Theft over honest toil.